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The normal shift of a rigid elliptical disk in a transversely isotropic solid

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Abstract

The paper addresses an elastostatic problem concerning an elliptical disk embedded in a transversely isotropic space. The disk is assumed to be absolutely rigid and in perfect contact with the medium. Then, the problem addressed consists of finding the elastic field in the medium when the disk is given a small shift along the direction perpendicular to its plane. Using the theory of two-dimensional Fourier transforms, the problem is reduced to a two-dimensional integral equation. Closed-form solution to this equation is obtained by using Ferrers–Dyson's and Galin's theorems. Explicit expressions are deduced for the displacements and stresses in the entire plane of the disk. Closed-form solution for the stress intensity factors near the edge of the disk is then extracted.

Keywords: Normal shift; Elliptical disk; Transversely isotropic solid

1. Introduction

Problems of cracks and inclusions in deformable solids have long been remaining as the subject matter of special interest to researchers from various areas such as mechanics, applied mathematics and materials science, because of the obvious practical applications of their solutions to the issues concerning strength degradation of solids containing cracks and inclusions. Much of the investigations done so far in this area are well-documented in the monographs by Panasyuk (1969), Galin (1976), Kassir and Sih (1975), Cherepanov (1979), Panasyuk et al. (1986), Mura (1987), Fabrikant (1991), Ting (1994), Phan-Thien and Kim (1994) and in the recent articles by the present author (Rahman (1999a,b,c; 2000a,b)). However, the bulk of the research in this direction is concerned with cracks and inclusions in isotropic materials. The related problems for anisotropic materials have been studied to a lesser extent, mainly because of the more complicated nature of their constitutive behaviors. However, in the case of transversely isotropic materials whose constitutive behaviors may be described by five independent elastic constants, solutions of a large number of crack and inclusion problems can be found.

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Of concern in the present article is the problem of determining the stress distribution in a transversely isotropic solid when a rigid elliptical disk embedded in it is given a small, constant shift along the direction perpendicular to its plane. It is assumed that the disk is in perfect contact with the medium. Using the theory of two-dimensional Fourier transforms, the problem is reduced to a two-dimensional integral equation. Closed-form solution of the integral equation is then deduced by using the theorems of Ferrers–Dyson and Galin – an extremely powerful tool in the theory of Newtonian potentials. A systematic use of this tool is capable of eliminating in toto the need to use the complicated apparatus of elliptical coordinates and elliptical harmonics – a topic first brought into the realm of mixed boundary value problems of elasticity by Lure (1939) (see also Lure (1964)), Galin (1947), and Green and Sneddon (1949). As we will see in the sequel, the use of Ferrers–Dyson's and Galin's theorems has certain unquestionable advantages in point of greater clearness and understanding of the physical quantities, being given in Cartesian form. Closed-form expressions for the displacements and stresses in the plane of the disk are then derived. It has been further shown that the amplification of the local stresses near the edge of the disk can be described by a coefficient similar to the mode-II stress intensity factor in linear fracture mechanics. Explicit expression for this factor is then extracted.

It is worth mentioning at this point that the corresponding problem for an isotropic solid for the case where the disk is of the shape of a penny was solved by Collins (1962). It was probably Kassir and Sih (1968) who first solved the problem for the case where the disk is of the shape of an ellipse. Following Green and Sneddon (1949) and their own earlier work (1966), these authors used elliptical coordinates and elliptical harmonics for the Laplace operator to solve the problem. The solution of our problem is found to be in perfect agreement with Collins' and Kassir and Sih's.

The results for the stress intensity factors obtained in the article are important in the sense that they can be used, in conjunction with a failure criterion, to predict the critical failure load and the initiation of crack propagation near the edge of the disk in a manner similar to that proposed by Panasyuk and Andreikiv (1967).

We begin by introducing the notation that we shall make use of. We shall denote the operator of the 2-dimensional Fourier transform by F_2 and its inverse by F_2^* so that F_2 is defined by

$$F_2[f(\bar{x});\bar{\alpha}] \equiv \tilde{f}(\bar{\alpha}) = \frac{1}{2\pi}\int_{R^2} f(\bar{x})\mathrm{e}^{-\mathrm{i}(\bar{\alpha}\cdot\bar{x})}\mathrm{d}\bar{x}$$

and F_2^* by

$$F_2^*[\tilde{f}(\bar{\alpha});\bar{x}] \equiv f(\bar{x}) = \frac{1}{2\pi}\int_{R^2} \tilde{f}(\bar{\alpha})\mathrm{e}^{-\mathrm{i}(\bar{\alpha}\cdot\bar{x})}\mathrm{d}\bar{\alpha}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, $\bar{x} = (x, y) \in R^2$, $\bar{\alpha} = (\alpha_1, \alpha_2) \in R^2$ and $\bar{\alpha}\cdot\bar{x} = \alpha_1 x + \alpha_2 y$ (Section 2.13 of Sneddon (1972)).

We write the convolution theorem in the form

$$F_2^*[\tilde{f}(\bar{\alpha})\tilde{g}(\bar{\alpha});\bar{x}] \equiv (f \circ g)(\bar{x}),$$

where $(f \circ g)(\bar{x})$ is defined by

$$(f \circ g)(\bar{x}) = \frac{1}{2\pi}\int_{R^2} f(\bar{x} - \bar{x}_0)g(\bar{x}_0)\mathrm{d}\bar{x}_0.$$

2. Potential representation for transversely isotropic bodies

In this section, we give a very brief outline of the potential solutions for solids characterized by transverse isotropy as developed by Elliott (1948, 1949). A convenient for our purpose summary of the basic results in this regard can be found in Rahman (1999a,b,c). Consider a transversely isotropic body occupying the space $(|x| < \infty, |y| < \infty, |z| < \infty)$. Further, assume that the axis of symmetry coincides with the z-axis. Under these assumptions, it was shown by Elliott (1948, 1949; see also Green and Zerna (1968)) that the equations of equilibrium and the compatibility equations for a transversely isotropic solid can be reduced to some Laplace-type equations involving three potential functions, χ_α $(\alpha = 1, 2, 3)$:

$$\left(\nabla_1^2 + s_\alpha \frac{\partial^2}{\partial z^2}\right)\chi_\alpha = 0, \tag{1}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad s_3 = \frac{2c_{44}}{c_{11} - c_{12}}, \tag{2}$$

and s_1, s_2 are the roots of the equation

$$c_{11}c_{44}s^2 + \{c_{13}(2c_{44} + c_{13}) - c_{11}c_{33}\}s + c_{33}c_{44} = 0, \tag{3}$$

and c_{ij} are the five elastic constants of the transversely isotropic solid. The physical foundations of these constants and their numerical values for a wide range of transversely isotropic materials are discussed at great length in Huntington (1964).

In terms of χ_α, the components of the displacement vector and stress tensor are given by

$$\begin{aligned}
&u = \frac{\partial}{\partial x}(\chi_1 + \chi_2) + \frac{\partial \chi_3}{\partial y}, \qquad v = \frac{\partial}{\partial y}(\chi_1 + \chi_2) - \frac{\partial \chi_3}{\partial x},\\
&w = k_1 \frac{\partial \chi_1}{\partial z} + k_2 \frac{\partial \chi_2}{\partial z}, \qquad \sigma_{zz} = \sum_{\alpha=1}^{2}(k_\alpha c_{33} - s_\alpha c_{13})\frac{\partial^2 \chi_\alpha}{\partial z^2},\\
&\sigma_{yz} = c_{44}\left\{-\frac{\partial^2 \chi_3}{\partial x \partial z} + \sum_{\alpha=1}^{2}(1 + k_\alpha)\frac{\partial^2 \chi_\alpha}{\partial y \partial z}\right\},\\
&\sigma_{zx} = c_{44}\left\{\frac{\partial^2 \chi_3}{\partial y \partial z} + \sum_{\alpha=1}^{2}(1 + k_\alpha)\frac{\partial^2 \chi_\alpha}{\partial x \partial z}\right\},
\end{aligned} \tag{4}$$

where

$$k_\alpha = \frac{c_{11}s_\alpha - c_{44}}{c_{13} + c_{44}} = \frac{(c_{13} + c_{44})s_\alpha}{c_{33} - c_{44}s_\alpha}, \quad \alpha = 1, 2. \tag{5}$$

The remaining stress components are not listed above, because we shall not need them in the subsequent analysis. Using the argument that for real materials the strain energy density function should be always positive definite, Lekhnitskii (1963) showed that the roots of the Eq. (3) may be either real (with the same sign) or complex conjugates, but they can never be purely imaginary. Furthermore, in order to ensure that the displacements and the stresses are single-valued, it is necessary to adopt a convention, in the event s_1, s_2 are negative or complex conjugates, on which branch of the multi-valued functions $\sqrt{s_1}$, $\sqrt{s_2}$ should be taken. In this article, we choose to take those branches, which have positive real parts.

3. Formulation of the problem

Consider an unbounded transversely isotropic medium in the interior of which there is an elliptical disk which is assumed to be perfectly rigid. Complete bonding is assumed to exist between the medium and the disk. We introduce a Cartesian coordinate system $O\,x\,y\,z$ such that the plane of disk coincides with the plane $z = 0$ and the elliptical bonded contact region Ω between the disk and the medium is given by the relations: $z = 0$, $\forall(x, y) \in x^2/a^2 + y^2/b^2 - 1 \leqslant 0 (a > b)$. We denote by $\widetilde{\Omega}$ the compliment of Ω in the plane $z = 0$. Furthermore, the upper and lower faces of the disk will be denoted by Ω_+ and Ω_-, respectively, so that $\Omega = \Omega_+ \cup \Omega_-$, and $\widetilde{\Omega} = \widetilde{\Omega}_+ \cup \widetilde{\Omega}_-$. The disk is given a small constant shift w_0 normal to its plane, i.e. the plane $z = 0$. [1] It is clear that the normal displacement $w(x, y, z)$ in the resulting elastic field will be an even function in z. Hence, the potential functions χ_α $(\alpha = 1, 2, 3)$ are odd in z, and so are the quantities $u(x, y, z)$, $v(x, y, z)$, $\sigma_{zz}(x, y, z)$. It therefore follows that all the quantities that are odd in z must vanish on the plane $z = 0$. In view of these observations, we can restrict the analysis to the upper half-space $z \geqslant 0$ only, in which case the following boundary conditions must be satisfied:

$$
\begin{aligned}
u(x, y, 0) &= 0, \qquad (x, y) \in \Omega_+ \cup \widetilde{\Omega}_+, \\
v(x, y, 0) &= 0, \qquad (x, y) \in \Omega_+ \cup \widetilde{\Omega}_+, \\
w(x, y, 0) &= w_0, \qquad (x, y) \in \Omega_+, \\
\sigma_{zz}(x, y, 0) &= 0, \qquad (x, y) \in \widetilde{\Omega}_+.
\end{aligned}
\tag{6}
$$

Besides, the solution of the problem must satisfy the regularity condition at infinity to ensure a decaying elastic field. More specifically, it is required that the displacement field be of $O(R^{-1})$ as $R = \sqrt{x^2 + y^2 + z^2} \to \infty$. Furthermore, in order to ensure that the solution of the problem be unique, it must satisfy the edge condition which states that the elastic energy stored in any neighborhood of the edge of the disk should be finite. It can be shown that this amounts to require that in the neighborhood of the edge, none of the field quantities grow more rapidly than $r^{-1+\varepsilon} (r = \sqrt{x^2 + y^2})$ with $\varepsilon > 0$ as $r \to 0$. Strictly speaking, it is not necessary to know a priori, the exact value of ε but only its lower bound, which is greater than zero, in order to derive a unique solution of the equilibrium equations. In many instances, however, it is convenient to have a prior knowledge of ε, which can be gained by using different methods, for instance, by the method of Hartranft and Sih (1969).

4. Solution

A suitable solution of Eq. (1) satisfying the regularity condition at infinity is given by

$$
\chi_j(x, y, z) = F_2^*[A_j(\bar{\alpha}) e^{-i\bar{\alpha}\cdot\bar{x} - m_j z}; \bar{x}], \tag{7}
$$

where $m_j = \sqrt{(\alpha_1^2 + \alpha_2^2)/s_j}$, and A_j $(j = 1, 2, 3)$ are some unknown functions to be determined using the boundary conditions of the problems.

With Eq. (7), we obtain the following expressions:

$$
\begin{aligned}
\tilde{u}(\alpha_1, \alpha_2, 0) &= -i(\alpha_1 A_1 + \alpha_1 A_2 + \alpha_2 A_3), \\
\tilde{v}(\alpha_1, \alpha_2, 0) &= -i(\alpha_2 A_1 + \alpha_2 A_2 - \alpha_1 A_3), \\
\tilde{w}(\alpha_1, \alpha_2, 0) &= -k_1 m_1 A_1 - k_2 m_2 A_2, \\
\tilde{\sigma}_{xz}(\alpha_1, \alpha_2, 0) &= i c_{44}[(1 + k_1)\alpha_1 m_1 A_1 + \alpha_2 m_3 A_3 + (1 + k_2)\alpha_1 m_2 A_2], \\
\tilde{\sigma}_{yz}(\alpha_1, \alpha_2, 0) &= i c_{44}[(1 + k_1)\alpha_2 m_1 A_1 + (1 + k_2)\alpha_2 m_2 A_2 - \alpha_1 m_3 A_3], \\
\tilde{\sigma}_{zz}(\alpha_1, \alpha_2, 0) &= (k_1 c_{33} - s_1 c_{13}) m_1^2 A_1 + (k_2 c_{33} - s_2 c_{13}) m_2^2 A_2.
\end{aligned}
\tag{8}
$$

[1] The amount of the shift must be small enough to justify the use of the equations of linearized elasticity.

We introduce the notations

$$p_1(x,y)=\sigma_{xz}(x,y,0),\qquad p_2(x,y,0)=\sigma_{yz}(x,y,0),\qquad p_3(x,y)=\sigma_{zz}(x,y,0).$$

Then, using Eq. (8) and invoking the first two boundary conditions in Eq. (6), we deduce that

$$\begin{aligned}\tilde{w}(\alpha_1,\alpha_2,0)&=\Gamma_1 c_{44}^{-1}\frac{\tilde{p}_3(\alpha_1,\alpha_2)}{\sqrt{\alpha_1^2+\alpha_2^2}},\\ \tilde{p}_1(\alpha_1,\alpha_2)&=\mathrm{i}\Gamma_2\frac{\alpha_1\tilde{p}_3(\alpha_1,\alpha_2)}{\sqrt{\alpha_1^2+\alpha_2^2}},\\ \tilde{p}_2(\alpha_1,\alpha_2)&=\mathrm{i}\Gamma_2\frac{\alpha_2\tilde{p}_3(\alpha_1,\alpha_2)}{\sqrt{\alpha_1^2+\alpha_2^2}},\end{aligned}\tag{9}$$

where

$$\Gamma_1=\frac{c_{44}\sqrt{s_1s_2}\left(k_2\sqrt{s_1}-k_1\sqrt{s_2}\right)}{c_{33}(k_1s_2-k_2s_1)},\qquad \Gamma_2=\frac{c_{44}\lfloor(1+k_1)s_2\sqrt{s_1}-(1+k_2)s_1\sqrt{s_2}\rfloor}{c_{33}(k_1s_2-k_2s_1)}.\tag{10}$$

Using the convolution theorem for Fourier transforms, from the Eq. (9), we have

$$\begin{aligned}w(x,y,0)&=\frac{\Gamma_1}{2\pi c_{44}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{p_3(x_0,y_0)}{R}\,\mathrm{d}x_0\,\mathrm{d}y_0,\\ p_1(x,y)&=\frac{-\Gamma_2}{2\pi}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{p_3(x_0,y_0)}{R}\,\mathrm{d}x_0\,\mathrm{d}y_0,\\ p_2(x,y)&=\frac{-\Gamma_2}{2\pi}\frac{\partial}{\partial y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{p_3(x_0,y_0)}{R}\,\mathrm{d}x_0\,\mathrm{d}y_0,\end{aligned}\tag{11}$$

where $R=\sqrt{(x-x_0)^2+(y-y_0)^2}$.

Using Eq. (1) in Eq. (11) and invoking the third and fourth boundary conditions in Eq. (6), we arrive at the following two-dimensional integral equation to determine the unknown interfacial normal stresses:

$$\int_{\Omega_+}\int\frac{p_3(x_0,y_0)}{R}\,\mathrm{d}x_0\,\mathrm{d}y_0=\frac{2\pi c_{44}}{\Gamma_1}w_0,\quad (x,y)\in\Omega_+.\tag{12}$$

By analogy, Eq. (12) represents the Newtonian potential of an elliptical disc whose mass distribution is characterized by the surface mass density $p_3(x,y)$. Therefore, the entire rich arsenal of the theory of the Newtonian potential of an elliptical disk, in particular, the theorems of Ferrers–Dyson (Dyson (1891)) and Galin (1947) can be employed to solve the integral equation (12). A convenient synopsis of Ferrers–Dyson's and Galin's theorems, their inter-relationships and some further extensions are given in Rahman (1999b, 2000c). Using these works, we represent the solution of Eq. (12) in the following form:

$$p_3(x,y)=\frac{A}{l(x,y)},\tag{13}$$

where A is an unknown constant to be determined and

$$l(x,y)=\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}.$$

Putting Eq. (13) into the Eq. (12), we have

$$A\int_{\Omega_+}\int\frac{\mathrm{d}x_0\,\mathrm{d}y_0}{l(x_0,y_0)R}=\frac{2\pi c_{44}}{\Gamma_1}w_0,\quad (x_1,x_2)\in\Omega_+.\tag{14}$$

As per Dyson's theorem (Dyson (1891), Rahman (2000c)), we have

$$\int_{\Omega_+}\int \frac{\mathrm{d}x_0\,\mathrm{d}y_0}{l(x_0,y_0)R} = \pi ab I_{00}^{(\lambda)}, \tag{15}$$

where

$$I_{ij}^{(\lambda)} = \int_{\lambda}^{\infty} \frac{\mathrm{d}\psi}{\sqrt{(a^2+\psi)^{2i+1}(b^2+\psi)^{2j+1}\psi}}. \tag{16}$$

Eqs. (15) and (16) are valid for the exterior of the disk, but they are also valid for points lying in the interior of the disk, i.e. $(x,y)\in\Omega$, provided λ is assumed to be equal to zero.

Recurrence relations were developed for evaluating closed-form expressions for $I_{ij}^{(\lambda)}$ for all $i,j \geqslant 0$ (see Rahman (2000c). The starting value for the recurrence relations is $I_{00}^{(\lambda)}$ which is evaluated in closed form (Gradshteyn and Ryzhik, 1994):

$$I_{00}^{(\lambda)} = \frac{2}{a}F(\beta,e), \tag{17}$$

where $F(\beta,e)$ is the incomplete elliptic integral of the first kind, $e=\sqrt{1-(b/a)^2}$ is the eccentricity of the ellipse and

$$\beta = \cot^{-1}\frac{\sqrt{\lambda}}{a}, \tag{18}$$

where

$$\lambda = \frac{x^2+y^2-a^2-b^2+\sqrt{D}}{2} \tag{19}$$

and

$$D = \left(x^2+y^2-a^2-b^2\right)^2 + 4a^2b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right). \tag{20}$$

As stated above, for $(x,y)\in\Omega$, $\lambda=0$ and hence, by Eq. (18), $\beta=\pi/2$. Therefore, for this case, Eq. (17) reduces to

$$I_{00}^{(0)} = \frac{2}{a}K(e), \tag{21}$$

where $K(e)$ is the complete elliptic integral of the first kind. Putting Eq. (21) into Eqs. (15) and (14), we find that the constant A is given by

$$A = \frac{w_0 c_{44}}{bK(e)\Gamma_1}. \tag{22}$$

Now, putting Eq. (22) into Eq. (13), we obtain the formula for normal stresses in the region occupied by the disk:

$$p_3(x,y) = c_{44}\frac{w_0}{bK(e)\Gamma_1}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-1/2}, \qquad (x,y)\in\Omega_+. \tag{23}$$

On the other hand, the normal displacement outside the disk is found by putting Eq. (23) into the first equation in Eq. (11) with the result

$$w(x,y,0) = \frac{w_0}{2\pi b K(e)} \int_{\Omega_+}\int \frac{\mathrm{d}x_0\,\mathrm{d}y_0}{l(x_0,y_0)R}, \quad (x,y)\in\widetilde{\Omega}_+. \tag{24}$$

Using the Eqs. (15) and (17), Eq. (24) is cast in the final form:

$$w(x,y,0) = \frac{w_0}{2\pi b K(e)} I_{00}^{(\lambda)} = \frac{F(\beta,e)}{K(e)} w_0, \quad (x,y)\in\widetilde{\Omega}_+. \tag{25}$$

Putting Eq. (23) into the second and third equations in Eq. (11), we derive the following formulae for the shearing stresses

$$p_i(x,y) = \left\{\begin{array}{ll} 0, & (x,y)\in\Omega_+, \\ -c_{44}\Gamma_3 \frac{w_0 a}{2K(e)} \frac{\partial I_{00}^{(\lambda)}}{\partial x_i} = & (x,y)\in\widetilde{\Omega}_+ \\ c_{44}\Gamma_3 \frac{w_0 a}{2K(e)\sqrt{(a^2+\lambda)(b^2+\lambda)\lambda}} \frac{\partial\lambda}{\partial x_i}, & (x,y)\in\widetilde{\Omega}_+ \end{array}\right\}, \quad (i=1,2), \tag{26}$$

where $x_1 = x$, $x_2 = y$ and

$$\Gamma_3 = \frac{\Gamma_2}{\Gamma_1} = \frac{(1+k_1)s_2\sqrt{s_1} - (1+k_2)s_1\sqrt{s_2}}{\sqrt{s_1 s_2}\left(k_2\sqrt{s_1} - k_1\sqrt{s_2}\right)}. \tag{27}$$

From Eq. (19), we have

$$\frac{\partial\lambda}{\partial x_i} = \frac{2x_i\left(a_{3-i}^2 + \lambda\right)}{\sqrt{D}}, \quad (x,y)\in\widetilde{\Omega}, \tag{28}$$

where $a_1 = a$, $a_2 = b$.

In view of Eq. (28), Eq. (26) is cast in the following final form:

$$p_i(x,y) = \left\{\begin{array}{ll} 0, & (x,y)\in\Omega_+, \\ c_{44}\Gamma_3 \frac{x_i w_0 a_1}{K(e)} \sqrt{\frac{a_{3-i}^2+\lambda}{\lambda D\left(a_i^2+\lambda\right)}}, & (i \text{ not summed}),\ (x,y)\in\widetilde{\Omega}_+ \end{array}\right\}, \quad (i=1,2). \tag{29}$$

Thus, the final solution of the problem is as follows:

$$\begin{aligned} w(x,y,0) &= \frac{F(\beta,e)}{K(e)} w_0, \quad (x,y)\in\widetilde{\Omega}_\pm, \\ p_i(x,y) &= \left\{\begin{array}{ll} 0, & (x,y)\in\Omega_\pm, \\ c_{44}\Gamma_3 \frac{x_i w_0 a_1}{K(e)} \sqrt{\frac{a_{3-i}^2+\lambda}{\lambda D\left(a_i^2+\lambda\right)}}, & (x,y)\in\widetilde{\Omega}_\pm. \end{array}\right\}, \quad (i=1,2), \\ p_3(x,y) &= \pm\frac{c_{44}w_0}{bK(e)\Gamma_1}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-(1/2)}, \quad (x,y)\in\Omega_\pm. \end{aligned} \tag{30}$$

Finally, we calculate the force exerted by the elastic medium to oppose the displacement of the disk; we have

$$\begin{aligned} P &= \int_{\Omega_+}\int [\sigma_{zz}(x,y,+0) - \sigma_{zz}(x,y,-0)]\mathrm{d}x\,\mathrm{d}y, \\ &= \frac{2c_{44}w_0}{bK(e)\Gamma_1}\int_{\Omega_+}\int \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-(1/2)} \mathrm{d}x\,\mathrm{d}y, \\ &= \frac{4\pi c_{44} a w_0}{K(e)\Gamma_1}. \end{aligned} \tag{31}$$

In the limit as $b \to a$, Eqs. (30) and (31) reduce to the solution for a circular disk, namely,

$$\begin{aligned} w(r,\theta,0) &= \frac{2w_0}{\pi}\sin^{-1}\frac{a}{r}, \quad r \geqslant a, \; 0 \leqslant \theta \leqslant 2\pi, \\ \tau(r,\theta) &= p_1(x,y)\cos\theta + p_2(x,y)\sin\theta = \left\{ \begin{array}{ll} 0, & 0 \leqslant r \leqslant a, \; 0 \leqslant \theta \leqslant 2\pi, \\ \frac{2c_{44}\Gamma_3 w_0}{\pi r\sqrt{(r/a)^2-1}}, & r > a, \; 0 \leqslant \theta \leqslant 2\pi. \end{array} \right\}, \\ p_3(r,\theta) &= \pm\frac{2c_{44}w_0}{\pi\Gamma_1\sqrt{a^2-r^2}}, \quad 0 \leqslant r < a, \; 0 \leqslant \theta \leqslant 2\pi, \\ P &= \frac{8c_{44}aw_0}{\Gamma_1}, \end{aligned} \tag{32}$$

where r, θ are the polar coordinates.

For the case of an isotropic material, we have (Lekhnitskii (1963))

$$c_{11} = c_{33} = \frac{2(1-v)\mu}{1-2v}, \quad c_{12} = c_{13} = \frac{2v\mu}{1-2v}, \qquad c_{44} = \mu, \tag{33}$$

where μ, v are respectively the shear modulus and Poisson's ratio. With Eq. (33), from Eqs. (3) and (5), we obtain

$$s_1 = s_2 = s_3 = 1. \tag{34}$$

Putting Eq. (34) into the equations for Γ_1 and Γ_2 (equations (10)), we see that they reduce to some indeterminacies. In order to overcome this difficulty, we assume that

$$s_1 = 1 + \mathrm{i}\varepsilon, \quad s_2 = 1 - \mathrm{i}\varepsilon, \qquad 0 < \varepsilon \ll 1. \tag{35}$$

Then, with Eq. (35), it can be shown that

$$\begin{aligned} k_{1,2} &= 1 \pm 2\mathrm{i}\varepsilon(1-v) + O(\varepsilon^2), \qquad \Gamma_1 = \frac{-(3-4v)}{4(1-v)} + O(\varepsilon), \\ \Gamma_2 &= \frac{1-2v}{2(1-v)} + O(\varepsilon), \qquad \Gamma_3 = \frac{-2(1-2v)}{3-4v} + O(\varepsilon). \end{aligned} \tag{36}$$

The solution of the problem corresponding to an isotropic solid can be recovered from Eqs. (30) and (31) by using Eqs. (33), (34) and (36) and then allowing $\varepsilon \to 0$ in the resulting expressions with the result:

$$\begin{aligned} w(x,y,0) &= \frac{F(\beta,e)}{K(e)}w_0, \quad (x,y) \in \widetilde{\Omega}_\pm, \\ p_i(x,y) &= \left\{ \begin{array}{ll} 0, & (x,y) \in \Omega_\pm, \\ \frac{-2(1-2v)\mu w_0 a_1 x_i}{(3-4v)K(e)}\sqrt{\frac{a_{3-i}^2+\lambda}{\lambda D(a_i^2+\lambda)}}, & (x,y) \in \widetilde{\Omega}_\pm. \end{array} \right\}, \quad (i = 1,2), \\ p_3(x,y) &= \mp\frac{4(1-v)\mu w_0}{b(3-4v)K(e)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-(1/2)}, \quad (x,y) \in \Omega_\pm, \\ P &= \frac{-16\pi(1-v)\mu w_0 a}{(3-4v)K(e)}. \end{aligned} \tag{37}$$

Eq. (37) is in agreement with Kassir and Sih (1968) if their results are cast in Cartesian coordinates.

Finally, the solution corresponding to a circular disk in an isotropic medium can be deduced by letting $b \to a$ in Eq. (37) with the result

$$
\begin{aligned}
w(r,\theta,0) &= \frac{2w_0}{\pi}\sin^{-1}\frac{a}{r}, \quad r \geqslant a, \;\; 0 \leqslant \theta \leqslant 2\pi, \\
\tau(r,\theta) &= \left\{ \begin{array}{ll} 0, & 0 \leqslant r \leqslant a, \;\; 0 \leqslant \theta \leqslant 2\pi, \\ \frac{-4(1-2v)\mu w_0}{\pi r(3-4v)\sqrt{(r/a)^2-1}}, & r > a, \;\; 0 \leqslant \theta \leqslant 2\pi. \end{array} \right\}, \\
p_3(r,\theta) &= \mp \frac{8(1-v)\mu w_0}{\pi(3-4v)\sqrt{a^2-r^2}}, \quad 0 \leqslant r < a, \;\; 0 \leqslant \theta \leqslant 2\pi, \\
P &= \frac{-32\mu(1-v)a w_0}{3-4v},
\end{aligned} \tag{38}
$$

which is in complete agreement with Collins (1962).

To gain an appreciation of the elegance and simplicity of the present method of solution, the reader is urged to compare it with that employed by Kassir and Sih (1966, 1968), which makes use of the complicated apparatus of elliptical coordinates and elliptical harmonics for the Laplace operator. The reader would notice that the use of the present approach requires only a modicum of knowledge in elementary calculus, yet the results obtained by this method, being given in Cartesian coordinates, are much clearer and easier to interpret than those given in elliptical coordinates by Kassir and Sih.

5. The behavior of the stress field near the disk edge

The analytical results obtained in the previous section are useful in analyzing the mechanics of fracture initiation at the edge of the disk. To this end, the behavior of the stress field in the immediate vicinity of the edge of the disk needs to be investigated. To effect this, we construct a curve by displacing the contour of the ellipse by a very small amount ρ along the outward normal (Fig. 1a). From elementary geometry, it follows that the slope of the normal is

$$\tan\alpha = \frac{a}{b}\tan\phi, \tag{39}$$

where ϕ is the parametric angle of the ellipse and α is the angle between the normal and the x-axis (Fig. 1a). Then, the equation of the displaced curve is given by the relations:

$$x = a\cos\phi + \rho\cos\alpha, \qquad y = b\sin\phi + \rho\sin\alpha. \tag{40}$$

The components of the shearing stress-vector, corresponding to the plane $z = 0$ along the normal and tangential directions at an arbitrary point of the contour of the disk (Fig. 1b) can be computed using the formulae:

$$
\begin{aligned}
p_n(x,y) &= p_1(x,y)n_1 + p_2(x,y)n_2, \\
p_t(x,y) &= -p_1(x,y)n_2 + p_2(x,y)n_1,
\end{aligned} \tag{41}
$$

where n_1 and n_2 are the directional cosines of the normal; they are given by the equations

$$n_1 = \frac{b\cos\phi}{\sqrt{\Pi(\phi)}}, \qquad n_2 = \frac{a\sin\phi}{\sqrt{\Pi(\phi)}}, \tag{42}$$

where

$$\Pi(\phi) = a^2\sin^2\phi + b^2\cos^2\phi.$$

Now, putting Eq. (40) into the second equation in Eq. (30) and expanding the resulting expressions in ρ/a, we obtain for smaller ρ/a, the following expressions:

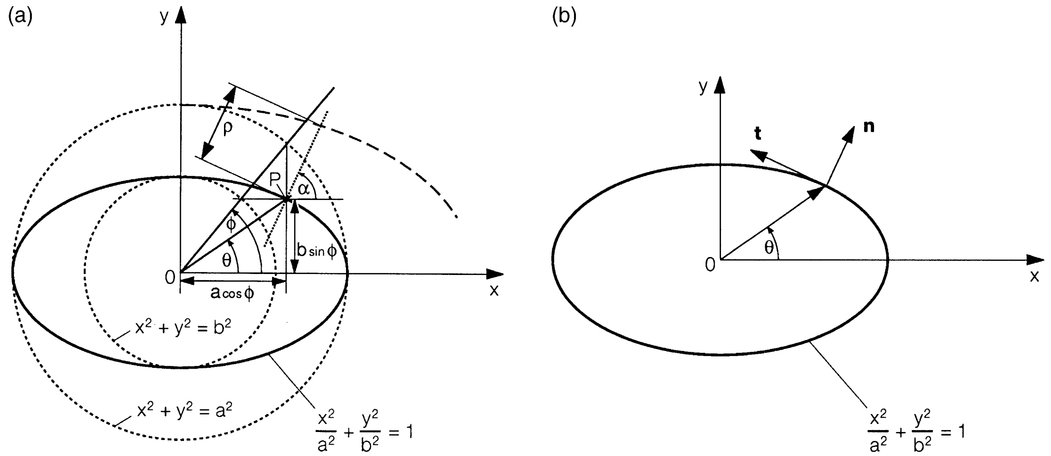

Fig. 1. Derivation of the asymptotic representation of the stress field near the edge of the disk.

$$
\begin{aligned}
p_1(x,y) &= \frac{\Lambda_1}{\sqrt{2\rho}} + O\left(\sqrt{\rho/a}\right), \quad (x,y) \in \widetilde{\Omega}_\pm, \\
p_2(x,y) &= \frac{\Lambda_2}{\sqrt{2\rho}} + O\left(\sqrt{\rho/a}\right), \quad (x,y) \in \widetilde{\Omega}_\pm,
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
\Lambda_1 &= \sqrt{\frac{a}{b}}\frac{c_{44}\Gamma_3 w_0 b\cos\phi}{K(e)\Pi^{3/4}(\phi)}, \\
\Lambda_2 &= \sqrt{\frac{a}{b}}\frac{c_{44}\Gamma_3 w_0 a\sin\phi}{K(e)\Pi^{3/4}(\phi)}.
\end{aligned}
\tag{44}
$$

With Eqs. (43) and (44) and using Eqs. (41) and (42), we have

$$
\begin{aligned}
p_3(x,y) &= 0, \quad (x,y) \in \widetilde{\Omega}_\pm, \\
p_n(x,y) &= \sqrt{\frac{a}{b}}\frac{c_{44}\Gamma_3 w_0}{\sqrt{2\rho}K(e)\Pi^{1/4}(\phi)} + O\left(\sqrt{\rho/a}\right), \quad (x,y) \in \widetilde{\Omega}_\pm, \\
p_t(x,y) &= O\left(\sqrt{\rho/a}\right), \quad (x,y) \in \widetilde{\Omega}_\pm.
\end{aligned}
\tag{45}
$$

From Eq. (45), we observe that the material in the immediate vicinity of the disk experiences an edge-sliding kinematic movement. According to the Griffith–Irwin theory of fracture, this mode of fracture can be best described by the corresponding stress intensity factor, which may be extracted from the equation:

$$
K_{\mathrm{II}} = \lim_{\rho\to 0}\sqrt{2\rho}\,p_n(x,y). \tag{46}
$$

Putting Eq. (45) into Eq. (46), we thus obtain

$$
K_{\mathrm{II}} = \sqrt{\frac{a}{b}}\frac{c_{44}\Gamma_3 w_0}{K(e)\Pi^{1/4}(\phi)}. \tag{47}
$$

The other two stress intensity factors, viz. K_{I} and K_{III} are, of course, zero.

For an isotropic material, Eq. (47) takes the form

$$K_{\mathrm{II}} = -\sqrt{\frac{a}{b}}\frac{2\mu(1-2v)w_0}{(3-4v)K(e)\Pi^{1/4}(\phi)},$$

which is in perfect agreement with the result obtained by Kassir and Sih (1968).

However, formula (47) should be cast in terms of the polar angle θ (Fig. 1b) as pointed out by Fabrikant (1987). Fabrikant's suggestion was bitterly criticized by a number of writers, e.g. Kassir and Sih (1988), and Zhang and Mai (1988). However, a recent look at the issue by Nuller et al. (1998) has confirmed the correctness of Fabrikant's suggestion.

We note that the angles ϕ and θ are related by the equations

$$\cos\phi = \frac{b\cos\theta}{\sqrt{\Pi(\theta)}}, \quad \sin\phi = \frac{a\sin\theta}{\sqrt{\Pi(\theta)}}. \tag{48}$$

With Eq. (48), we have

$$\begin{aligned} &K_{\mathrm{I}} = K_{\mathrm{III}} = 0, \\ &K_{\mathrm{II}} = \sqrt{\frac{a}{b}}\frac{c_{44}\Gamma_3 w_0}{K(e)}\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^4\sin^2\theta + b^4\cos^2\theta}\right)^{1/4}. \end{aligned} \tag{49}$$

It is claimed that the solution given by Eqs. (30) and (49) is new.

On noting that

$$\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^4\sin^2\theta + b^4\cos^2\theta}\right)^{1/4} = [abR(\theta)]^{-(1/6)},$$

($R(\theta)$ is the radius of curvature of the elliptical curve at the point corresponding to the polar angle θ), Eq. (49) can be represented as

$$\begin{aligned} &K_{\mathrm{I}} = K_{\mathrm{III}} = 0, \\ &K_{\mathrm{II}} = \frac{c_{44}\Gamma_3 w_0 a^{1/3}}{K(e) b^{2/3} R^{1/6}(\theta)}. \end{aligned} \tag{50}$$

For an isotropic material, Eqs. (49) and (50) take the form

$$\begin{aligned} K_{\mathrm{I}} &= K_{\mathrm{III}} = 0, \\ K_{\mathrm{II}} &= -\sqrt{\frac{a}{b}}\frac{2\mu(1-2v)w_0}{(3-4v)K(e)}\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^4\sin^2\theta + b^4\cos^2\theta}\right)^{1/4}, \\ &= \frac{-2\mu(1-2v)w_0 a^{1/3}}{(3-4v)K(e)b^{2/3}R^{1/6}(\theta)}. \end{aligned} \tag{51}$$

6. Conclusion

In the present article, the problem of determining the stress distribution in a transversely isotropic solid caused by an axial displacement of an embedded rigid elliptical disk is considered. Using the theory of two-dimensional Fourier transforms, the problem is reduced to a two-dimensional integral equation whose closed-form solution is deduced by using the theorems of Ferrers–Dyson and Galin. Explicit expressions for the stress intensity factors at the edge of the disk are then extracted from this solution. These results might be used with a suitable failure criterion to interpret the mechanism of fracture initiation near the edge of the disk. The method of solution developed in the article can be used for a wide variety of mixed boundary value problems of elasticity theory concerning elliptical cracks and inclusions. Of further interest are the corresponding problem for a system of coplanar elliptical disks and extension of the present results to elastodynamics. Another problem for possible future research is the corresponding problem of a disk in the form of an elliptical ring formed by two confocal ellipses. A solution of a problem of this kind in two-dimensional elasticity, with the aid of Muskhelishvili's complex variable approach, has been recently reported by Kawakubo and Hirashima (1997). However, it is not clear how their results can be generalized to three dimensions, especially since no analytical expressions are available in the literature for the Newtonian potential of an elliptical ring.

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